

# Factorizations of Matrices Over Projective-free Rings

H. Chen<sup>\*</sup>, H. Kose<sup>†</sup>, Y. Kurtulmaz<sup>‡</sup>

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## Abstract

An element of a ring  $R$  is called strongly  $J^\#$ -clean provided that it can be written as the sum of an idempotent and an element in  $J^\#(R)$  that commute. We characterize, in this article, the strongly  $J^\#$ -cleanness of matrices over projective-free rings. These extend many known results on strongly clean matrices over commutative local rings.

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## 1 Introduction

Let  $R$  be a ring with an identity. We say that  $x \in R$  is strongly clean provided that there exists an idempotent  $e \in R$  such that  $x - e \in U(R)$  and  $ex = xe$ . A ring  $R$  is strongly clean in case every element in  $R$  is strongly clean (cf. [9-10]). In [2, Theorem 12], Borooah, Diesl, and Dorsey provide the following characterization: Given a commutative local ring  $R$  and a monic polynomial  $h \in R[t]$  of degree  $n$ , the following are equivalent: (1)  $h$  has an *SRC* factorization in  $R[t]$ ; (2) every  $\varphi \in M_n(R)$  which satisfies  $h$  is strongly clean. It is demonstrated in [6, Example 3.1.7] that statement (1) of the above can not be weakened from *SRC* factorization to *SR* factorization. The purpose of this paper is to investigate a subclass of strongly clean rings which behave like such ones but can be characterized by a kind of *SR* factorizations, and so get more explicit factorizations for many class of matrices over projective-free rings.

Let  $J(R)$  be the Jacobson radical of  $R$ . Set

$$J^\#(R) = \{x \in R \mid \exists n \in \mathbb{N} \text{ such that } x^n \in J(R)\}.$$

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<sup>\*</sup>Department of Mathematics, Hangzhou Normal University, Hangzhou, 310036, People's Republic of China, e-mail: huanyinchen@yahoo.cn

<sup>†</sup>Department of Mathematics, Ahi Evran University, Kirsehir, Turkey, handankose@gmail.com

<sup>‡</sup>Department of Mathematics, Bilkent University, Ankara, Turkey, yosum@fen.bilkent.edu.tr

For instance, let  $R = M_2(\mathbb{Z}_2)$ . Then

$$J^\#(R) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\},$$

while  $J(R) = 0$ . Thus,  $J^\#(R)$  and  $J(R)$  are distinct in general. We say that an element  $a \in R$  is strongly  $J^\#$ -clean provided that there exists an idempotent  $e \in R$  such that  $a - e \in J^\#(R)$  and  $ea = ae$ . If  $R$  is a commutative ring, then  $a \in R$  is strongly  $J^\#$ -clean if and only if  $a \in R$  is strongly  $J$ -clean (cf. [3]). But they behave different for matrices over commutative rings. A Jordan-Chevalley decomposition of  $n \times n$  matrix  $A$  over an algebraically closed field (e.g., the field of complex numbers), then  $A$  is an expression of it as a sum:  $A = E + W$ , where  $E$  is semisimple,  $W$  is nilpotent, and  $E$  and  $W$  commute. The Jordan-Chevalley decomposition is extensively studied in Lie theory and operator algebra. As a corollary, we will completely determine when an  $n \times n$  matrix over a field is the sum of an idempotent matrix and a nilpotent matrix that commute. Thus, the strongly  $J^\#$ -clean factorizations of matrices over rings is also an analog of that of Jordan-Chevalley decompositions for matrices over fields.

We characterize, in this article, the strongly  $J^\#$ -cleanness of matrices over projective-free rings. Here, a commutative ring  $R$  is projective-free provided that every finitely generated projective  $R$ -module is free. For instances, every commutative local ring, every commutative semi-local ring, every principal ideal domain, every Bézout domain (e.g., the ring of all algebraic integers) and the ring  $R[x]$  of all polynomials over a principal domain  $R$  are all projective-free. We will show that strongly  $J^\#$ -clean matrices over projective-free rings are completely determined by a kind of “ $SC$ ”-factorizations of the characteristic polynomials. These extend many known results on strongly clean matrices to such new factorizations of matrices over projective-free rings (cf. [1-2] and [5]).

Throughout, all rings with an identity and all modules are unitary modules. Let  $f(t) \in R[t]$ . We say that  $f(t)$  is a monic polynomial of degree  $n$  if  $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$  where  $a_{n-1}, \dots, a_1, a_0 \in R$ . We always use  $U(R)$  to denote the set of all units in a ring  $R$ . If  $\varphi \in M_n(R)$ , we use  $\chi(\varphi)$  to stand for the characteristic polynomial  $\det(tI_n - \varphi)$ .

## 2 Full Matrices Over Projective-free Rings

Let  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$ . It is directly verified that  $A \in M_2(\mathbb{Z}_2)$  is not strongly  $J^\#$ -clean, though  $A$  is strongly clean. It is hard to determine strongly cleanness even for matrices over the integers, but completely different situation is in the strongly  $J^\#$ -clean case. The aim of this section is to characterize a single strongly  $J^\#$ -clean  $n \times n$  matrix

over projective-free rings. Let  $M$  be a left  $R$ -module. We denote the endomorphism ring of  $M$  by  $\text{end}(M)$ .

**Lemma 2.1** *Let  $M$  be a left  $R$ -module, and let  $E = \text{end}(M)$ , and let  $\alpha \in E$ . Then the following are equivalent:*

- (1)  $\alpha \in E$  is strongly  $J^\#$ -clean.
- (2) There exists a left  $R$ -module decomposition  $M = P \oplus Q$  where  $P$  and  $Q$  are  $\alpha$ -invariant, and  $\alpha|_P \in J^\#(\text{end}(P))$  and  $(1_M - \alpha)|_Q \in J^\#(\text{end}(Q))$ .

**Proof** (1)  $\Rightarrow$  (2) Since  $\alpha$  is strongly  $J^\#$ -clean in  $E$ , there exists an idempotent  $\pi \in E$  and a  $u \in J^\#(E)$  such that  $\alpha = (1 - \pi) + u$  and  $\pi u = u\pi$ . Thus,  $\pi\alpha = \pi u \in J^\#(\pi E \pi)$ . Further,  $1 - \alpha = \pi + (-u)$ , and so  $(1 - \pi)(1 - \alpha) = (1 - \pi)(-u) \in J^\#((1 - \pi)E(1 - \pi))$ . Set  $P = M\pi$  and  $Q = M(1 - \pi)$ . Then  $M = P \oplus Q$ . As  $\alpha\pi = \pi\alpha$ , we see that  $P$  and  $Q$  are  $\alpha$ -invariant. As  $\alpha\pi \in J^\#(\pi E \pi)$ , we can find some  $t \in \mathbb{N}$  such that  $(\alpha\pi)^t \in J(\pi E \pi)$ . Let  $\gamma \in \text{end}(P)$ . For any  $x \in M$ , it is easy to see that  $(x)\pi(1_P - \gamma(\alpha|_P)^t) = (x)\pi(\pi - (\pi\bar{\gamma}\pi)(\pi\alpha\pi)^t)$  where  $\bar{\gamma}: M \rightarrow M$  given by  $(m)\bar{\gamma} = (m)\pi\gamma$  for any  $m \in M$ . Hence,  $1_P - \gamma(\alpha|_P)^t \in \text{aut}(P)$ . Hence  $(\alpha|_P)^t \in J(\text{end}(P))$ . This implies that  $\alpha|_P \in J^\#(\text{end}(P))$ . Likewise, we verify that  $(1 - \alpha)|_Q \in J^\#(\text{end}(Q))$ .

(2)  $\Rightarrow$  (1) For any  $\lambda \in \text{end}(Q)$ , we construct an  $R$ -homomorphism  $\bar{\lambda} \in \text{end}(M)$  given by  $(p + q)\bar{\lambda} = (q)\lambda$ . By hypothesis,  $\alpha|_P \in J^\#(\text{end}(P))$  and  $(1_M - \alpha)|_Q \in J^\#(\text{end}(Q))$ . Thus,  $\alpha = \overline{1_Q} + \overline{\alpha|_P} - \overline{(1_M - \alpha)|_Q}$ . As  $P$  and  $Q$  are  $\alpha$ -invariant, we see that  $\alpha\overline{1_Q} = \overline{1_Q}\alpha$ . In addition,  $\overline{1_Q} \in \text{end}(M)$  is an idempotent. As  $(\overline{\alpha|_P})(\overline{(1_M - \alpha)|_Q}) = 0 = (\overline{(1_M - \alpha)|_Q})(\overline{\alpha|_P})$ , we show that  $\overline{\alpha|_P} - \overline{(1_M - \alpha)|_Q} \in J^\#(\text{end}(M))$ , as required.  $\square$

**Lemma 2.2** *Let  $R$  be a ring, and let  $M$  be a left  $R$ -module. Suppose that  $x, y, a, b \in \text{end}(M)$  such that  $xa + yb = 1_M, xy = yx = 0, ay = ya$  and  $xb = bx$ . Then  $M = \ker(x) \oplus \ker(y)$  as left  $R$ -modules.*

**Proof** Straightforward. (cf. [6, Lemma 3.2.6]).  $\square$

**Lemma 2.3** *Let  $R$  be a commutative ring, and let  $\varphi \in M_n(R)$ . Then the following are equivalent:*

- (1)  $\varphi \in J^\#(M_n(R))$ .
- (2)  $\chi(\varphi) \equiv t^n \pmod{J(R)}$ , i.e.,  $\chi(\varphi) - t^n \in J(R)[t]$ .
- (3) There exists a monic polynomial  $h \in R[t]$  such that  $h \equiv t^{\text{deg}h} \pmod{J(R)}$  for which  $h(\varphi) = 0$ .

**Proof** (1)  $\Rightarrow$  (2) Since  $\varphi \in J^\#(M_n(R))$ , there exists some  $m \in \mathbb{N}$  such that  $\varphi^m \in J(M_n(R))$ . As  $J(M_n(R)) = M_n(J(R))$ , we get  $\overline{\varphi} \in N(M_n(R/J(R)))$ . In view of [6, Proposition 3.5.4],  $\chi(\overline{\varphi}) \equiv t^n \pmod{N(R/J(R))}$ . Write  $\chi(\varphi) = t^n + a_1 t^{n-1} + \cdots + a_n$ . Then  $\chi(\overline{\varphi}) = t^n + \overline{a_1} t^{n-1} + \cdots + \overline{a_n}$ . We infer that each  $a_i^{m_i} + J(R) = 0 + J(R)$  where  $m_i \in \mathbb{N}$ . This implies that  $a_i \in J^\#(R)$ . That is,  $\chi(\varphi) \equiv t^n \pmod{J^\#(R)}$ . Obviously,  $J(R) \subseteq J^\#(R)$ . For any  $x \in J^\#(R)$ , then there exists some  $m \in \mathbb{N}$  such that  $x^m \in J(R)$ . For any maximal ideal  $M$  of  $R$ ,  $M$  is prime, and so  $x \in M$ . This implies that  $x \in J(R)$ ; hence,  $J^\#(R) \subseteq J(R)$ . Therefore  $J^\#(R) = J(R)$ , as required.

(2)  $\Rightarrow$  (3) Choose  $h = \chi(\varphi)$ . Then  $h \equiv t^{\deg h} \pmod{J(R)}$ . In light of the Cayley-Hamilton Theorem,  $h(\varphi) = 0$ , as required.

(3)  $\Rightarrow$  (1) By hypothesis, there exists a monic polynomial  $h \in R[t]$  such that  $h \equiv t^{\deg h} \pmod{J(R)}$  for which  $h(\varphi) = 0$ . Write  $h = t^n + a_1 t^{n-1} + \cdots + a_n$ . Choose  $\overline{h} = t^n + \overline{a_1} t^{n-1} + \cdots + \overline{a_n} \in (R/J(R))[t]$ . Then  $\overline{h} \equiv t^n \pmod{N(R/J(R))}$  for which  $\overline{h}(\overline{\varphi}) = 0$ . According to [6, Proposition 3.5.4], there exists some  $m \in \mathbb{N}$  such that  $(\overline{\varphi})^m = \overline{0}$  over  $R/J(R)$ . Therefore  $\varphi^m \in M_n(J(R))$ , and so  $\varphi \in J^\#(M_n(R))$ .  $\square$

**Definition 2.4** For  $r \in R$ , define

$$\mathbb{J}_r = \{f \in R[t] \mid f \text{ monic, and } f \equiv (t - r)^{\deg f} \pmod{J^\#(R)}\}.$$

**Lemma 2.5** Let  $R$  be a projective-free ring, let  $\varphi \in M_n(R)$ , and let  $h \in R[t]$  be a monic polynomial of degree  $n$ . If  $h(\varphi) = 0$  and there exists a factorization  $h = h_0 h_1$  such that  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ , then  $\varphi$  is strongly  $J^\#$ -clean.

**Proof** Suppose that  $h = h_0 h_1$  where  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ . Write  $h_0 = t^p + a_1 t^{p-1} + \cdots + a_p$  and  $h_1 = (t - 1)^q + b_1 t^{q-1} + \cdots + b_q$ . Then each  $a_i, b_j \in J^\#(R)$ . Since  $R$  is commutative, we get each  $a_i, b_j \in J(R)$ . Thus,  $\overline{h_0} = t^p$  and  $\overline{h_1} = (t - 1)^q$  in  $(R/J(R))[t]$ . Hence,  $(\overline{h_0}, \overline{h_1}) = \overline{1}$ . In virtue of [6, Lemma 3.5.10], we have some  $u_0, u_1 \in R[t]$  such that  $u_0 h_0 + u_1 h_1 = 1$ . Then  $u_0(\varphi) h_0(\varphi) + u_1(\varphi) h_1(\varphi) = 1_{nR}$ . By hypothesis,  $h(\varphi) = h_0(\varphi) h_1(\varphi) = h_1(\varphi) h_0(\varphi) = 0$ . Clearly,  $u_0(\varphi) h_1(\varphi) = h_1(\varphi) u_0(\varphi)$  and  $h_0(\varphi) u_1(\varphi) = u_1(\varphi) h_0(\varphi)$ . In light of Lemma 2.2,  $nR = \ker(h_0(\varphi)) \oplus \ker(h_1(\varphi))$ . As  $h_0 t = t h_0$  and  $h_1 t = t h_1$ , we see that  $h_0(\varphi) \varphi = \varphi h_0(\varphi)$  and  $h_1(\varphi) \varphi = \varphi h_1(\varphi)$ , and so  $\ker(h_0(\varphi))$  and  $\ker(h_1(\varphi))$  are both  $\varphi$ -invariant. It is easy to verify that  $h_0(\varphi|_{\ker(h_0(\varphi))}) = 0$ . Since  $h_0 \in \mathbb{J}_0$ , we see that  $h_0 \equiv t^{\deg h_0} \pmod{J^\#(R)}$ ; hence,  $\varphi|_{\ker(h_0(\varphi))} \in J^\#(\text{end}(\ker h_0(\varphi)))$ .

It is easy to verify that  $h_1(\varphi|_{\ker(h_1(\varphi))}) = 0$ . Set  $g(u) = (-1)^{\deg h_1} h_1(1 - u)$ . Then  $g((1 - \varphi)|_{\ker(h_1(\varphi))}) = 0$ . Since  $h_1 \in \mathbb{J}_1$ , we see that  $h_1 \equiv (t - 1)^{\deg h_1} \pmod{J^\#(R)}$ . Hence,  $g(u) \equiv (-1)^{\deg h_1} (-u)^{\deg g} \pmod{J(R)}$ . This implies that  $g \in \mathbb{J}_0$ . By virtue of Lemma 2.3,  $(1 - \varphi)|_{\ker(h_1(\varphi))} \in J^\#(\text{end}(\ker(h_1(\varphi))))$ . According to Lemma 2.1,  $\varphi \in M_n(R)$  is strongly  $J^\#$ -clean.  $\square$

The matrix

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R)$$

is called the companion matrix  $C_h$  of  $h$ , where  $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$ .

**Theorem 2.6** *Let  $R$  be a projective-free ring and let  $h \in R[t]$  be a monic polynomial of degree  $n$ . Then the following are equivalent:*

- (1) *Every  $\varphi \in M_n(R)$  with  $\chi(\varphi) = h$  is strongly  $J^\#$ -clean.*
- (2) *The companion matrix  $C_h$  of  $h$  is strongly  $J^\#$ -clean.*
- (3) *There exists a factorization  $h = h_0h_1$  such that  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ .*

**Proof** (1)  $\Rightarrow$  (2) Write  $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$ . Choose

$$C_h = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R).$$

Then  $\chi(C_h) = h$ . By hypothesis,  $C_h \in M_n(R)$  is strongly  $J^\#$ -clean.

(2)  $\Rightarrow$  (3) In view of Lemma 2.1, there exists a decomposition  $nR = A \oplus B$  such that  $A$  and  $B$  are  $\varphi$ -invariant,  $\varphi|_A \in J^\#(\text{end}_R(A))$  and  $(1 - \varphi)|_B \in J^\#(\text{end}_R(B))$ . Since  $R$  is a projective-free ring, there exist  $p, q \in \mathbb{N}$  such that  $A \cong pR$  and  $B \cong qR$ . Regarding  $\text{end}_R(A)$  as  $M_p(R)$ , we see that  $\varphi|_A \in J^\#(M_p(R))$ . By virtue of Lemma 2.3,  $\chi(\varphi|_A) \equiv t^p \pmod{J^\#(R)}$ . Thus  $\chi(\varphi|_A) \in \mathbb{J}_0$ . Analogously,  $(1 - \varphi)|_B \in J^\#(M_q(R))$ . It follows from Lemma 2.3 that  $\chi((1 - \varphi)|_B) \equiv t^q \pmod{J^\#(R)}$ . This implies that  $\det(\lambda I_q - (1 - \varphi)|_B) \equiv \lambda^q \pmod{J^\#(R)}$ . Hence,  $\det((1 - \lambda)I_q - \varphi|_B) \equiv (-\lambda)^q \pmod{J^\#(R)}$ . Set  $t = 1 - \lambda$ . Then  $\det(tI_q - \varphi|_B) \equiv (t - 1)^q \pmod{J^\#(R)}$ . Therefore we get  $\chi(\varphi|_B) \equiv (t - 1)^q \pmod{J^\#(R)}$ . We infer that  $\chi(\varphi|_B) \in \mathbb{J}_1$ . Clearly,  $\chi(\varphi) = \chi(\varphi|_A)\chi(\varphi|_B)$ . Choose  $h_0 = \chi(\varphi|_A)$  and  $h_1 = \chi(\varphi|_B)$ . Then there exists a factorization  $h = h_0h_1$  such that  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ , as desired.

(3)  $\Rightarrow$  (1) For every  $\varphi \in M_n(R)$  with  $\chi(\varphi) = h$ , it follows by the Cayley-Hamilton Theorem that  $h(\varphi) = 0$ . Therefore  $\varphi$  is strongly  $J^\#$ -clean by Lemma 2.5.  $\square$

**Corollary 2.7** *Let  $F$  be a field, and let  $A \in M_n(F)$ . Then the following are equivalent:*

- (1)  *$A$  is the sum of an idempotent matrix and a nilpotent matrix that commute.*

(2)  $\chi(A) = t^s(t-1)^t$  for some  $s, t \geq 0$ .

**Proof** As  $J(M_n(F)) = 0$ , we see that a  $n \times n$  matrix contains in  $J^\#(M_n(F))$  if and only if  $A$  is a nilpotent matrix. So  $A \in M_n(F)$  is strongly  $J^\#$ -clean if and only if  $A$  is the sum of an idempotent matrix and a nilpotent matrix that commute. By virtue of Theorem 2.6, we see that  $A \in M_n(F)$  is the sum of an idempotent matrix and a nilpotent matrix that commute if and only if  $\chi(A) = h_0 h_1$ , where  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ . Clearly,  $h_0 \in \mathbb{J}_0$  if and only if  $h_0 \equiv t^{\deg h_0} \pmod{J^\#(F)}$ . But  $J^\#(F) = 0$ , and so  $h_0 = t^s$ , where  $s = \deg h_0$ . Likewise,  $h_1 = (t-1)^t$ , where  $t = \deg h_1$ . Therefore we complete the proof.  $\square$

For matrices over integers, we have a similar situation. As  $J(M_n(\mathbb{Z})) = 0$ , we see that an  $n \times n$  matrix contains in  $J^\#(M_n(\mathbb{Z}))$  if and only if it is a nilpotent matrix. Likewise, we show that  $A \in M_n(\mathbb{Z})$  is the sum of an idempotent matrix and a nilpotent matrix that commute if and only if  $\chi(A) = t^s(t-1)^t$  for some  $s, t \geq 0$ . For instance,

choose  $A = \begin{pmatrix} -2 & 2 & -1 \\ -4 & 4 & -2 \\ -1 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{Z})$ . Then  $\chi(A) = t(t-1)^2$ . Thus,  $A$  is the sum of an idempotent matrix and an nilpotent matrix that commute. In fact, we have a corresponding factorization  $A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$ .

**Corollary 2.8** *Let  $R$  be a projective-free ring, and let  $\varphi \in M_2(R)$ . Then  $\varphi$  is strongly  $J^\#$ -clean if and only if*

- (1)  $\chi(\varphi) \equiv t^2 \pmod{J(R)}$ ; or
- (2)  $\chi(\varphi) \equiv (t-1)^2 \pmod{J(R)}$ ; or
- (3)  $\chi(\varphi)$  has a root in  $J(R)$  and a root in  $1 + J(R)$ .

**Proof** Suppose that  $\varphi$  is strongly  $J^\#$ -clean. By virtue of Theorem 2.6, there exists a factorization  $\chi(\varphi) = h_0 h_1$  such that  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ .

Case I.  $\deg(h_0) = 2$  and  $\deg(h_1) = 0$ . Then  $h_0 = \chi(\varphi) = t^2 - \text{tr}(\varphi)t + \det(\varphi)$  and  $h_1 = 1$ . As  $h_0 \in \mathbb{J}_0$ , it follows from Lemma 2.3 that  $\varphi \in J^\#(M_2(R))$  or  $\chi(\varphi) \equiv t^2 \pmod{J(R)}$ .

Case II.  $\deg(h_0) = 1$  and  $\deg(h_1) = 1$ . Then  $h_0 = t - \alpha$  and  $h_1 = t - \beta$ . Since  $R$  is commutative,  $J^\#(R) = J(R)$ . As  $h_0 \in \mathbb{J}_0$ , we see that  $h_0 \equiv t \pmod{J(R)}$ , and then  $\alpha \in J(R)$ . As  $h_1 \in \mathbb{J}_1$ , we see that  $h_1 \equiv t - 1 \pmod{J(R)}$ , and then  $\beta \in 1 + J(R)$ . Therefore  $\chi(\varphi)$  has a root in  $J(R)$  and a root in  $1 + J(R)$ .

Case III.  $\deg(h_0) = 0$  and  $\deg(h_1) = 2$ . Then  $h_1(t) = \det(tI_2 - \varphi) \equiv (t-1)^2 \pmod{J(R)}$ . Set  $u = 1 - t$ . Then  $\det(uI_2 - (I_2 - \varphi)) \equiv u^2 \pmod{J(R)}$ . According to Lemma 2.3,  $I_2 - \varphi \in J^\#(M_2(R))$  or  $\chi(\varphi) \equiv (t-1)^2 \pmod{J(R)}$ .

We will suffice to show the converse. If  $\chi(\varphi) \equiv t^2 \pmod{J(R)}$  or  $\chi(\varphi) \equiv (t-1)^2 \pmod{J(R)}$ , then  $\varphi \in J^\#(M_2(R))$  or  $I_2 - \varphi \in J^\#(M_2(R))$ . This implies that  $\varphi$  is strongly  $J^\#$ -clean. Otherwise,  $\varphi, I_2 - \varphi \notin J(M_2(R))$ . In addition,  $\chi(\varphi)$  has a root in  $J(R)$  and a root in  $1 + J(R)$ . According to [4, Theorem 16.4.31],  $\varphi$  is strongly  $J$ -clean, and therefore it is strongly  $J^\#$ -clean.  $\square$

Choose  $A = \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{1} & \bar{3} \end{pmatrix} \in M_2(\mathbb{Z}_4)$ . It is easy to check that  $A, I_2 - A \in M_2(\mathbb{Z}_4)$  are not nilpotent. But  $\chi(A) = t^2 + t + 2$  has a root  $\bar{2} \in J(\mathbb{Z}_4)$  and a root  $\bar{1} \in 1 + J(\mathbb{Z}_4)$ . As  $J(\mathbb{Z}_4) = \{\bar{0}, \bar{2}\}$  is nil, we know that every matrix in  $J^\#(M_2(\mathbb{Z}_4))$  is nilpotent. It follows from Corollary 2.8 that  $A$  is the sum of an idempotent matrix and a nilpotent matrix that commute. Let  $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n\}$ , and let  $A = \begin{pmatrix} 1 & 1 \\ \frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$ . Then  $J(\mathbb{Z}_{(2)}) = \{\frac{2m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n\}$ . As  $\chi(A) = t^2 - t + \frac{2}{9}$  has a root  $\frac{1}{3} \in 1 + J(\mathbb{Z}_{(2)})$  and a root  $\frac{2}{3} \in J(\mathbb{Z}_{(2)})$ . In light of Corollary 2.8,  $A$  is strongly  $J$ -clean.

**Corollary 2.9** *Let  $R$  be a projective-free ring, and let  $f(t) = t^2 + at + b \in R[t]$  be degree 2 polynomial with  $1 + a \in J(R), b \notin J(R)$ . Then the following are equivalent:*

- (1) *Every  $\varphi \in M_2(R)$  with  $\chi(\varphi) = f(t)$  is strongly  $J^\#$ -clean.*
- (2) *There exist  $r_1 \in J(R)$  and  $r_2 \in 1 + J(R)$  such that  $f(r_i) = 0$ .*
- (3) *There exists  $r \in J(R)$  such that  $f(r) = 0$ .*

**Proof** (1)  $\Rightarrow$  (2) Since every  $\varphi \in M_2(R)$  with  $\chi(\varphi) = f(t)$  is strongly  $J^\#$ -clean, it follows by Corollary 2.8 that  $f(t) = (t - r_1)(t - r_2)$  with  $r_1 \in J(R), r_2 \in 1 + J(R)$ .

(2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1) As  $r^2 + ar + b = 0$ , we see that  $f(t) = (t - r)(t + a + r)$ . Clearly,  $t - r \in \mathbb{J}_0$ . As  $1 + a + r \in J(R)$ , we see that  $t + a + r \in \mathbb{J}_1$ . According to Theorem 2.6, we complete the proof.  $\square$

Let  $\varphi$  be a  $3 \times 3$  matrix over a commutative ring  $R$ . Set  $\text{mid}(\varphi) = \det(I_3 - \varphi) - 1 + \text{tr}(\varphi) + \det(\varphi)$ .

**Corollary 2.10** *Let  $R$  be a projective-free ring, and let  $\varphi \in M_3(R)$ . Then  $\varphi$  is strongly  $J^\#$ -clean if and only if*

- (1)  $\chi(\varphi) \equiv t^3 \pmod{J(R)}$ ; or
- (2)  $\chi(\varphi) \equiv (t-1)^3 \pmod{J(R)}$ ; or
- (3)  $\chi(\varphi)$  has a root in  $1 + J(R), \text{tr}(\varphi) \in 1 + J(R), \text{mid}(\varphi) \in J(R), \det(\varphi) \in J(R)$ ; or
- (4)  $\chi(\varphi)$  has a root in  $J(R), \text{tr}(\varphi) \in 2 + J(R), \text{mid}(\varphi) \in 1 + J(R), \det(\varphi) \in J(R)$ .

**Proof** Suppose that  $\varphi$  is strongly  $J^\#$ -clean. By virtue of Theorem 2.6, there exists a factorization  $\chi(\varphi) = h_0 h_1$  such that  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ .

Case I.  $\deg(h_0) = 3$  and  $\deg(h_1) = 0$ . Then  $h_0 = \chi(\varphi)$  and  $h_1 = 1$ . As  $h_0 \in \mathbb{J}_0$ , it follows from Lemma 2.3 that  $\varphi \in J^\#(M_3(R))$ .

Case II.  $\deg(h_0) = 0$  and  $\deg(h_1) = 3$ . Then  $h_1(t) = \det(tI_3 - \varphi) \equiv (t - 1)^3 \pmod{J(R)}$ . Set  $u = 1 - t$ . Then  $\det(uI_3 - (I_3 - \varphi)) \equiv u^3 \pmod{J(R)}$ . According to Lemma 2.3,  $I_3 - \varphi \in J^\#(M_3(R))$ .

Case III.  $\deg(h_0) = 2$  and  $\deg(h_1) = 1$ . Then  $h_0 = t^2 + at + b$  and  $h_1 = t - \alpha$ . As  $h_0 \in \mathbb{J}_0$ , we see that  $h_0 \equiv t^2 \pmod{J(R)}$ ; hence,  $a, b \in J(R)$ . As  $h_1 \in \mathbb{J}_1$ , we see that  $h_1 \equiv t - 1 \pmod{J(R)}$ ; hence,  $\alpha \in 1 + J(R)$ . We see that  $a - \alpha = -\text{tr}(\varphi)$ ,  $b - a\alpha = \text{mid}(\varphi)$  and  $-b\alpha = -\det(\varphi)$ . Therefore  $\text{tr}(\varphi) \in 1 + J(R)$ ,  $\text{mid}(\varphi) \in J(R)$  and  $\det(\varphi) \in J(R)$ .

Case IV.  $\deg(h_0) = 1$  and  $\deg(h_1) = 2$ . Then  $h_0 = t - \alpha$  and  $h_1 = t^2 + at + b$ . As  $h_0 \in \mathbb{J}_0$ , we see that  $h_0 \equiv t \pmod{J(R)}$ ; hence,  $\alpha \in J(R)$ . As  $h_1 \in \mathbb{J}_1$ , we see that  $h_1 \equiv (t - 1)^2 \pmod{J(R)}$ , and then  $a \in -2 + J(R)$  and  $b \in 1 + J(R)$ . Obviously,  $\chi(\varphi) = t^3 - \text{tr}(\varphi)t^2 + \text{mid}(\varphi)t - \det(\varphi)$ , and so  $a - \alpha = -\text{tr}(\varphi)$ ,  $b - a\alpha = \text{mid}(\varphi)$  and  $-b\alpha = -\det(\varphi)$ . Therefore  $\text{tr}(\varphi) \in 2 + J(R)$ ,  $\text{mid}(\varphi) \in 1 + J(R)$  and  $\det(\varphi) \in J(R)$ .

Conversely, if  $\chi(\varphi) \equiv t^3 \pmod{J(R)}$  or  $\chi(\varphi) \equiv (t - 1)^3 \pmod{J(R)}$ , then  $\varphi \in J^\#(M_3(R))$  or  $I_3 - \varphi \in J^\#(M_3(R))$ . Hence,  $\varphi$  is strongly  $J^\#$ -clean. Suppose  $\chi(\varphi)$  has a root  $\alpha \in 1 + J(R)$  and  $\text{tr}(\varphi) \in 1 + J(R)$ ,  $\det(\varphi) \in J(R)$ . Then  $\chi(\varphi) = (t^2 + at + b)(t - \alpha)$  for some  $a, b \in R$ . This implies that  $a - \alpha = -\text{tr}(\varphi)$ ,  $-b\alpha = -\det(\varphi)$ . Hence,  $a, b \in J(R)$ . Let  $h_0 = t^2 + at + b$  and  $h_1 = t - \alpha$ . Then  $\chi(\varphi) = h_0 h_1$  where  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ . According to Theorem 2.6,  $\varphi$  is strongly  $J^\#$ -clean.

Suppose  $\chi(\varphi)$  has a root  $\alpha \in J(R)$  and  $\text{tr}(\varphi) \in 2 + J(R)$ ,  $\text{mid}(\varphi) \in 1 + J(R)$  and  $\det(\varphi) \in J(R)$ . Then  $\chi(\varphi) = (t - \alpha)(t^2 + at + b)$  for some  $a, b \in R$ . This implies that  $a - \alpha = -\text{tr}(\varphi)$ ,  $b - a\alpha = \text{mid}(\varphi)$ . Hence,  $a \in -2 + J(R)$ ,  $b \in 1 + J(R)$ . Let  $h_0 = t - \alpha$  and  $h_1 = t^2 + at + b$ . Then  $\chi(\varphi) = h_0 h_1$  where  $h_0 \in \mathbb{J}_0$  and  $h_1 \in \mathbb{J}_1$ . According to Theorem 2.6,  $\varphi$  is strongly  $J^\#$ -clean, and we are done.  $\square$

### 3 Matrices Over Power Series Rings

The purpose of this section is to extend the preceding discussion to matrices over power series rings. We use  $R[[x]]$  to stand for the ring of all power series over  $R$ . Let  $A(x) = (a_{ij}(x)) \in M_n(R[[x]])$ . We use  $A(0)$  to stand for  $(a_{ij}(0)) \in M_n(R)$ .

**Theorem 3.1** *Let  $R$  be a projective-free ring, and let  $A(x) \in M_2(R[[x]])$ . Then the following are equivalent:*

- (1)  $A(x) \in M_2(R[[x]])$  is strongly  $J^\#$ -clean.
- (2)  $A(0) \in M_2(R)$  is strongly  $J^\#$ -clean.



**Proof** (1)  $\Rightarrow$  (2) Since  $A(x)$  is strongly  $J^\#$ -clean in  $M_2(R[[x]])$ , there exists an  $E(x) = E^2(x) \in M_2(R[[x]])$  and a  $U(x) \in J^\#(M_2(R[[x]]))$  such that  $A(x) = E(x) + U(x)$  and  $E(x)U(x) = U(x)E(x)$ . This implies that  $A(0) = E(0) + U(0)$  and  $E(0)U(0) = U(0)E(0)$  where  $E(0) = E^2(0) \in M_2(R)$  and  $U(0) \in J^\#(M_2(R))$ . As a result,  $A(0)$  is strongly  $J^\#$ -clean in  $M_2(R)$ .

(2)  $\Rightarrow$  (1) Construct a ring morphism  $\varphi : R[[x]] \rightarrow R, f(x) \mapsto f(0)$ . Then  $R \cong R[[x]]/\ker f$ , where  $\ker f = \{f(x) \mid f(0) = 0\} \subseteq J(R[[x]])$ . For any finitely generated projective  $R[[x]]$ -module  $P$ ,  $P \otimes_R (R[[x]]/\ker f)$  is a finitely generated projective  $R[[x]]/\ker f$ -module; hence it is free. Write  $P \otimes_R (R[[x]]/\ker f) \cong (R[[x]]/\ker f)^m$  for some  $m \in \mathbb{N}$ . Then  $P \otimes_R (R[[x]]/\ker f) \cong (R[[x]])^m \otimes_R (R[[x]]/\ker f)$ . That is,  $P/P(\ker f) \cong (R[[x]])^m / (R[[x]])^m (\ker f)$  with  $\ker f \subseteq J(R[[x]])$ . By Nakayama Theorem,  $P \cong (R[[x]])^m$  is free. Thus,  $R[[x]]$  is projective-free. Since  $A(0)$  is strongly  $J^\#$ -clean in  $M_2(R)$ , it follows from Corollary 2.8 that  $A(0) \in J^\#(M_2(R))$ , or  $I_2 - A(0) \in J^\#(M_2(R))$ , or the characteristic polynomial  $\chi(A(0)) = y^2 + \mu y + \lambda$  has a root  $\alpha \in 1 + J(R)$  and a root  $\beta \in J(R)$ . If  $A(0) \in J^\#(M_2(R))$ , then  $A(x) \in J^\#(M_2(R[[x]]))$ . If  $I_2 - A(0) \in J^\#(M_2(R))$ , then  $I_2 - A(x) \in J^\#(M_2(R[[x]]))$ . Otherwise, we write  $y = \sum_{i=0}^{\infty} b_i x^i$  and  $\chi(A(x)) = y^2 - \mu(x)y - \lambda(x)$ . Then  $y^2 = \sum_{i=0}^{\infty} c_i x^i$  where  $c_i = \sum_{k=0}^i b_k b_{i-k}$ . Let  $\mu(x) = \sum_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$  where  $\mu_0 = \mu$  and  $\lambda_0 = \lambda$ . Then,  $y^2 - \mu(x)y - \lambda(x) = 0$  holds in  $R[[x]]$  if the following equations are satisfied:

$$\begin{aligned} b_0^2 - b_0\mu_0 - \lambda_0 &= 0; \\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) - \lambda_1 &= 0; \\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) - \lambda_2 &= 0; \\ &\vdots \end{aligned}$$

Obviously,  $\mu_0 = \alpha + \beta \in U(R)$  and  $\alpha - \beta \in U(R)$ . Let  $b_0 = \alpha$ . Since  $R$  is commutative, there exists some  $b_1 \in R$  such that

$$b_0b_1 + b_1(b_0 - \mu_0) = \lambda_1 + b_0\mu_1.$$

Further, there exists some  $b_2 \in R$  such that

$$b_0b_2 + b_2(b_0 - \mu_0) = \lambda_2 - b_1^2 + b_0\mu_2 + b_1\mu_1.$$

By iteration of this process, we get  $b_3, b_4, \dots$ . Then  $y^2 - \mu(x)y - \lambda(x) = 0$  has a root  $y_0(x) \in 1 + J(R[[x]])$ . If  $b_0 = \beta \in J(R)$ , analogously, we show that  $y^2 - \mu(x)y - \lambda(x) = 0$  has a root  $y_1(x) \in J(R[[x]])$ . In light of Corollary 2.8, the result follows.  $\square$

**Corollary 3.2** *Let  $R$  be a projective-free ring, and let  $A(x) \in M_2(R[[x]]/(x^m))$  ( $m \geq 1$ ). Then the following are equivalent:*

- (1)  $A(x) \in M_2(R[[x]]/(x^m))$  is strongly  $J^\#$ -clean.
- (2)  $A(0) \in M_2(R)$  is strongly  $J^\#$ -clean.

**Proof** (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (1) Let  $\psi : R[[x]] \rightarrow R[[x]]/(x^m)$ ,  $\psi(f) = \bar{f}$ . Then it reduces a surjective ring homomorphism  $\psi^* : M_2(R[[x]]) \rightarrow M_2(R[[x]]/(x^m))$ . Hence, we have a  $B \in M_2(R[[x]])$  such that  $\psi^*(B(x)) = A(x)$ . According to Theorem 3.1, we complete the proof.  $\square$

**Example 3.3** Let  $R = \mathbb{Z}_4[x]/(x^2)$ , and let  $A(x) = \begin{pmatrix} \bar{2} & \bar{2} + \bar{2}x \\ \bar{2} + x & \bar{3} + \bar{3}x \end{pmatrix} \in M_2(R)$ . Obviously,  $\mathbb{Z}_4$  is a projective-free ring, and that  $R = \mathbb{Z}_4[[x]]/(x^2)$ . Since we have the strongly  $J^\#$ -clean decomposition  $A(0) = \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{2} & \bar{1} \end{pmatrix} + \begin{pmatrix} \bar{2} & \bar{0} \\ \bar{0} & \bar{2} \end{pmatrix}$  in  $M_2(\mathbb{Z}_4)$ , it follows by Corollary 3.2 that  $A(x) \in M_2(R)$  is strongly  $J^\#$ -clean.

**Theorem 3.4** *Let  $R$  be a projective-free ring, and let  $A(x) \in M_3(R[[x]])$ . Then the following are equivalent:*

- (1)  $A(x) \in M_3(R[[x]])$  is strongly  $J^\#$ -clean.
- (2)  $A(x) \in M_3(R[[x]]/(x^m))$  ( $m \geq 1$ ) is strongly  $J^\#$ -clean.
- (3)  $A(0) \in M_3(R)$  is strongly  $J^\#$ -clean.

**Proof** (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are clear.

(3)  $\Rightarrow$  (1) As  $A(0)$  is strongly  $J^\#$ -clean in  $M_3(R)$ , it follows from Corollary 2.10 that  $A(0) \in J^\#(M_3(R))$ , or  $I_3 - A(0) \in J^\#(M_3(R))$ , or  $\chi(A(0))$  has a root in  $J(R)$  and  $\text{tr}(A(0)) \in 2 + J(R)$ ,  $\text{mid}(A(0)) \in 1 + J(R)$ ,  $\det(A(0)) \in J(R)$ , or  $\chi(A(0))$  has a root in  $1 + J(R)$  and  $\text{tr}(A(0)) \in 1 + J(R)$ ,  $\text{mid}(A(0)) \in J(R)$ ,  $\det(A(0)) \in J(R)$ . If  $A(0) \in J^\#(M_3(R))$  or  $I_3 - A(0) \in J^\#(M_3(R))$ , then  $A(x) \in J^\#(M_3(R[[x]]))$  or  $I_3 - A(x) \in J^\#(M_3(R[[x]]))$ . Hence,  $A(x) \in M_3(R[[x]])$  is strongly  $J^\#$ -clean. Assume that  $\chi(A(0)) = t^3 - \mu t^2 - \lambda t - \gamma$  has a root  $\alpha \in J(R)$  and  $\text{tr}(A(0)) \in 2 + J(R)$ ,  $\text{mid}(A(0)) \in 1 + J(R)$ ,  $\det(A(0)) \in J(R)$ . Write  $y = \sum_{i=0}^{\infty} b_i x^i$ . Then  $y^2 = \sum_{i=0}^{\infty} c_i x^i$  where  $c_i = \sum_{k=0}^i b_k b_{i-k}$ . Further,  $y^3 = \sum_{i=0}^{\infty} d_i x^i$  where  $d_i = \sum_{k=0}^i b_k c_{i-k}$ . Let  $\mu(x) = \sum_{i=0}^{\infty} \mu_i x^i$ ,  $\lambda(x) = \sum_{i=0}^{\infty} \lambda_i x^i$ ,  $\gamma(x) = \sum_{i=0}^{\infty} \gamma_i x^i \in R[[x]]$  where  $\mu_0 = \mu$ ,  $\lambda_0 = \lambda$  and  $\gamma_0 = \gamma$ . Then,  $y^3 - \mu(x)y^2 - \lambda(x)y - \gamma(x) = 0$ .

$\lambda(x)y - \gamma(x) = 0$  holds in  $R[[x]]$  if the following equations are satisfied:

$$\begin{aligned} b_0^3 - b_0^2\mu_0 - b_0\lambda_0 - \gamma_0 &= 0; \\ (3b_0^2 - 2b_0\mu_0 - \lambda_0)b_1 &= \gamma_1 + b_0^2\mu_1 + b_0\lambda_1; \\ (3b_0^2 - 2b_0\mu_0 - \lambda_0)b_2 &= \gamma_2 + b_0^2\mu_2 + b_1^2\mu_0 + 2b_0b_1\mu_1 + b_0\lambda_2 + b_1\lambda_0 - 3b_0b_1^2; \\ &\vdots \end{aligned}$$

Let  $b_0 = \alpha \in J(R)$ . Obviously,  $\mu_0 = \text{tr} A(0) \in 2 + J(R)$  and  $\lambda_0 = -\text{mid} A(0) \in U(R)$ . Hence,  $3b_0^2 - 2b_0\mu_0 - \lambda_0 \in U(R)$ . Thus, we see that  $b_1 = (3b_0^2 - 2b_0\mu_0 - \lambda_0)^{-1}(\gamma_1 + b_0^2\mu_1 + b_0\lambda_1)$  and  $b_2 = (3b_0^2 - 2b_0\mu_0 - \lambda_0)^{-1}(\gamma_2 + b_0^2\mu_2 + b_1^2\mu_0 + 2b_0b_1\mu_1 + b_0\lambda_2 + b_1\lambda_0 - 3b_0b_1^2)$ . By iteration of this process, we get  $b_3, b_4, \dots$ . Then  $y^3 - \mu(x)y^2 - \lambda(x)y - \gamma(x) = 0$  has a root  $y_0(x) \in J(R[[x]])$ . It follows from  $\text{tr} A(0) \in 2 + J(R)$  that  $\text{tr} A(x) \in 2 + J(R[[x]])$ . Likewise,  $\text{mid} A(x) \in 1 + J(R[[x]])$ . According to Corollary 2.10,  $A(x) \in M_3(R[[x]])$  is strongly  $J^\#$ -clean.

Assume that  $\chi(A(0))$  has a root  $1 + \alpha \in J(R)$  and  $\text{tr}(A(0)) \in 1 + J(R)$ ,  $\text{mid}(A(0)) \in J(R)$ ,  $\det A(0) \in J(R)$ . Then  $\det(I_3 - A(0)) = 1 - \text{tr} A(0) + \text{mid} A(0) - \det A(0) \in J(R)$ . Set  $B(x) = I_3 - A(x)$ . Then  $\chi(B(0))$  has a root  $\alpha \in J(R)$  and  $\text{tr}(B(0)) \in 2 + J(R)$ ,  $\det B(0) \in J(R)$ . This implies that  $\text{mid} B(0) = \det A(0) - 1 + \text{tr} B(0) + \det B(0) \in 1 + J(R)$ . By the preceding discussion, we see that  $B(x) \in M_3(R[[x]])$  is strongly  $J^\#$ -clean, and then we are done.  $\square$

From this evidence above, we end this paper by asking the following question: Let  $R$  be a projective-free ring, and let  $A(x) \in M_n(R[[x]])$  ( $n \geq 4$ ). Do the strongly  $J^\#$ -cleanness of  $A(x) \in M_3(R[[x]])$  and  $A(0) \in M_3(R)$  coincide with each other?

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